TE-polarized waves guided by a lossless nonlinear three-layer structure

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We study TE-polarized electromagnetic waves guided by a three-layer structure consisting of a film surrounded by semi-infinite media. All three media are assumed to be lossless, nonmagnetic, isotropic, and exhibiting a local Kerr-like dielectric nonlinearity. We present general necessary and sufficient conditions for the existence of ''physical'' (real, nonnegative, bounded, and consistent with the dispersion relation) field intensities. As a physical consequence, the parameters $a_\nu, \overline{\epsilon}_\nu, n, E_0^2, k_0 d$ associated with realizable waves can be specified. To illustrate the procedure, analytical and numerical results for the allowed normalized thickness of the film and patterns of the field intensities as functions of the effective wave number and the intensity of the electric field at the lower surface of the nonlinear dielectric film are presented and the occurrence of singular field intensities is investigated. Finally, the particularization of the results to the three-layer structure containing a linear substrate and film and a nonlinear cladding is briefly discussed. [S1063-651X(98)13006-4]

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I. INTRODUCTION

In the past years, several papers [1–4] have been published concerning the propagation of the TE-polarized waves supported by a lossless isotropic nonlinear three-layer structures with the permittivity

$$\epsilon_\nu = \overline{\epsilon}_\nu + a_\nu |\vec{E}|^2, \tag{1}$$

where

$$\nu = \begin{cases} c, & z>d, \\ f, & 0<z<d, \\ s, & z<0 \end{cases}$$

and $\vec{E}$ denotes the electric field in the layers, assuming that $\overline{\epsilon}_\nu$ and a_ν are real constants. As a result, guided stationary TE waves are represented by the electric field

$$\vec{E} = \vec{e}_y E(n,z,\omega_0) e^{i(nk_0 x - \omega_0 t)}, \tag{2}$$

if the layers are homogeneous perpendicular to the z direction (see Fig. 1). $\vec{e}_y$ is the unit vector of the axis Oy, nk_0 denotes the = effective wave number, $k_0 = \omega_0 \sqrt{\mu_0 \epsilon_0}$ is the wave number of free space, and ω_0 is the (fixed) angular frequency of the wave. The real amplitude function $E = E(n,z,\omega_0)$ must be a solution to the equations [2,4]

$$\left(\frac{dE}{dz}\right)^2 - k_0^2\left(q_\nu^2 - \frac{a_\nu}{2}E^2\right)E^2 = k_0^2 C_\nu, \tag{3}$$

with the constants of integration C_ν to be determined by the boundary conditions and

$$q_\nu^2 = n^2 - \overline{\epsilon}_\nu, \quad \nu = s,f,c. \tag{4}$$

Equation (3) is solved by [4,5]

$$E_{\nu\pm}^2(n,z,\omega_0) = \frac{2}{a_\nu}\left[\wp(\omega_\nu \pm ik_0 z; g_{2\nu}, g_{3\nu}) + \frac{q_\nu^2}{3}\right], \tag{5}$$

where

$$\omega_\nu = \int_{(1/2)\, a_\nu E_\nu^2(n,0,\omega_0) - (1/3)\, q_\nu^2}^{\infty} \frac{dI}{\sqrt{4I^3 - g_{2\nu} I - g_{3\nu}}}, \quad \nu = s,f, \tag{6a}$$

$$\omega_c \pm ik_0 d = \int_{(1/2)\, a_c E_{c\pm}^2(n,d,\omega_0) - (1/3)\, q_c^2}^{\infty} \frac{dI}{\sqrt{4I^3 - g_{2c} I - g_{3c}}}, \tag{6b}$$

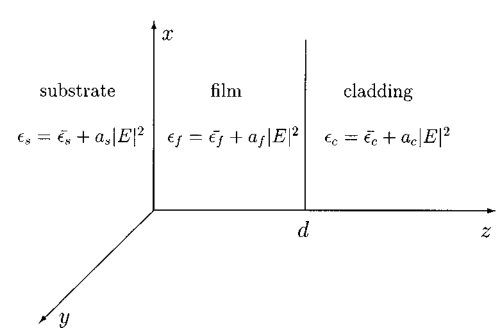

FIG. 1. Geometry considered in the paper. Three layers $\nu = s,f,c$ with permittivities $\epsilon_\nu = \overline{\epsilon}_\nu + a_\nu |\vec{E}|^2$ supporting stationary waves polarized in the y direction.

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$$g_{2\nu}=2(a_\nu C_\nu+\tfrac{2}{3}q_\nu^4), \tag{7}$$

$$g_{3\nu}=\tfrac{2}{3}q_\nu^2(a_\nu C_\nu+\tfrac{4}{9}q_\nu^4), \tag{8}$$

and $\wp(\omega_\nu \pm ik_0z;g_{2\nu},g_{3\nu})$ denotes Weierstrass's elliptic function with invariants $g_{2\nu}$ and $g_{3\nu}$. In consistency with Eq. (3), $\wp$ satisfies the equation

$$\left(\frac{d\wp(\omega_\nu \pm ik_0z)}{d(ik_0z)}\right)^2=\wp^3-g_{2\nu}\wp-g_{3\nu}$$
$$=(\wp-I_{1\nu})(\wp-I_{2\nu})(\wp-I_{3\nu}). \tag{9}$$

The discriminant $\Delta_\nu=g_{2\nu}^3-27g_{3\nu}^2$ of Weierstrass's function $\wp$ can be written as

$$\Delta_\nu=a_\nu^2C_\nu^2(2a_\nu C_\nu+q_\nu^4)$$
$$=16(I_{1\nu}-I_{2\nu})^2(I_{2\nu}-I_{3\nu})^2(I_{1\nu}-I_{3\nu})^2, \tag{10}$$

where

$$I_{1,3\nu}=\frac{q_\nu^2}{6}\pm\sqrt{\frac{a_\nu}{2}C_\nu+\frac{q_\nu^4}{4}}, \tag{11}$$

$$I_{2\nu}=-\frac{q_\nu^2}{3}.$$

As will be seen below, Δ_f and $I_{1,2,3f}$ are very useful quantities for the subsequent analysis. For the linear case ($a_\nu=0$), the conditions the physically satisfactory solutions to Eq. (3) must fulfill are well known [6]. The aim of the present paper is to infer from Eq. (5) the corresponding conditions for the nonlinear case. With respect to the dispersion relation (''mode condition'') in the linear case [6] it should be noted that this condition cannot be fulfilled for certain domains of values $n,\overline{\epsilon}_\nu$ irrespective the values of k_0d [7]. Below, we present the corresponding conditions of solvability (CS) of the dispersion relation (DR) in the nonlinear case. $E_{\nu\pm}^2(n,z,\omega_0)$, according to Eq. (5), must be real nonnegative and bounded for all z. To find the associated conditions (CRNB), for obvious reasons the well-known properties of $\wp$ [8] must be used. The CS can be derived in the same manner. To be ''physical'' the field intensities $E_{\nu\pm}^2$ must obey CRNB and DR (subject to CS). As a result, all these conditions and the allowed normalized thicknesses k_0d can be expressed in terms of Weierstrass's function $\wp$, its half-periods, and the associated quantities $\Delta_f,\omega_f,w_\pm,I_{1f},I_{2f},I_{3f}$. Since they are necessary and sufficient, CRNB, DR, and CS specify all tupels $\{a_\nu,\overline{\epsilon}_\nu,n,E^2(n,0,\omega_0),k_0d\}$ that are associated with physical solutions according to Eq. (5). Thus, the physical significance of CRNB, DR, and CS is the possibility to determine the realizable parameters $\{a_\nu,\overline{\epsilon}_\nu,n,E^2(n,0,\omega_0),k_0d\}$ corresponding to guided waves.

The paper is organized as follows. In Sec. II, we derive the CRNB. Section III contains solutions k_0d of the DR and conditions for its solvability (CS). Applications are given in Sec. IV. Finally, Sec. V summarizes the results. The Appendix contains mathematical details of Secs. II and III.

II. REAL, NONNEGATIVE, AND BOUNDED FIELD INTENSITIES

The electric field E must satisfy

$$E\to 0, \quad |z|\to\infty. \tag{12}$$

Equations (3) and (12) imply

$$C_s=C_c=0, \tag{13}$$

and, according to Eq. (10),

$$\Delta_s=\Delta_c=0. \tag{14}$$

Evaluating Eq. (5), using Eqs. (13) and (14), we obtain [9], subject to the condition

$$q_\nu^2\geqslant 0, \tag{15}$$

$$E_{\nu\pm}^2(z)=\frac{2q_\nu^2}{a_\nu\sin^2[\sqrt{q_\nu^2}(\omega_\nu\pm ik_0z)]}, \quad \nu=s,c. \tag{16}$$

If $q_\nu^2<0$, $E_{\nu\pm}^2(z)$ does not satisfy condition (12). Straightforward evaluation of Eq. (16) yields the necessary and sufficient conditions for $E_{\nu\pm}^2(z)$, $\nu=s,c$ to be real and nonnegative:

$$\mathrm{Re}\ \omega_\nu=\begin{cases}\pi l/\sqrt{q_\nu^2}, & a_\nu<0\\ (\tfrac{1}{2}\pi+\pi l)/\sqrt{q_\nu^2}, & a_\nu>0,\end{cases} \tag{17}$$

with $l\in\mathbb{Z}$ ($\mathbb{Z}$ denoting the set of whole numbers) and subject to $q_\nu^2\geqslant 0$. If $q_\nu^2=0$, only the case $a_\nu<0$ is possible with $l=0$ in Eq. (17).

Subject to Eqs. (15) and (17) Eqs. (16) represent bounded $E_{\nu\pm}^2(z),\nu=s,c$, if $a_\nu>0$. If $a_\nu<0$ only $E_{s-}^2(z)$ and $E_{c+}^2(z)$ are bounded, since $\mathrm{Im}\omega_\nu>0$ (cf. Appendix). Before deriving the CRNB for $E_{f\pm}^2(z)$, it is suitable to make use of the boundary conditions at the interfaces $z=0$ and $z=d$. Since both $E_{\nu\pm}^2$, $\nu=s,c$, and its derivatives with respect to z are continuous, we obtain [4]

$$\omega_s=\frac{1}{\sqrt{q_s^2}}\arcsin\sqrt{\frac{2q_s^2}{a_sE_0^2}}, \tag{18}$$

$$\omega_c\pm ik_0d=\frac{1}{\sqrt{q_c^2}}\arcsin\sqrt{\frac{2q_c^2}{a_cE_{c\pm}^2(d)}}, \tag{19}$$

with $E_0^2=E_s^2(n,0,\omega_0)>0$ and

$$C_f=E_0^2[\overline{\epsilon}_f-\overline{\epsilon}_s+\tfrac{1}{2}(a_f-a_s)E_0^2]. \tag{20}$$

Equations (18) and (19) must be consistent with Eqs. (17). Hence, we obtain, evaluating $\arcsin(\cdot)$ in Eqs. (18) and (19) [10], additionally to $q_\nu^2\geqslant 0$, the conditions

$$q_s^2>\frac{a_s}{2}E_0^2, \tag{21}$$

$$q_c^2 > \frac{a_c}{2} E_{c\pm}^2(d) \ , \tag{22}$$

where $E_{c\pm}^2(d)$ must be determined by Eq. (37) below. It should be noted that conditions (15), (21), and (22) are generalizations of the corresponding conditions in the linear case [6]. Turning to the CRNB for $E_{f\pm}^2$ and using properties of Weierstrass's function $\wp$ the following necessary and sufficient conditions for $E_{f\pm}^2$ being real can be derived [11]:

$$\text{Re } \omega_f = l\omega \qquad \text{for } \Delta_f > 0 \tag{23}$$

and

$$\text{Re } \omega_f = l\omega_2 \qquad \text{for } \Delta_f < 0 \tag{24}$$

must hold. ω, ω_2 denote the associated real half-periods of $\wp$ [8]. The case $\Delta_f = 0$ has been disregarded for reasons of simplicity. Subject to the conditions for the field intensities to be real (CR) Eqs. (21)–(24) ω_ν, $\nu = s,f,c$ can be written down explicitly (cf. Appendix). This implies that conditions (23) and (24) containing elliptic integrals, can be simplified considerably by using the roots I_{1f}, I_{2f}, I_{3f} defined in Eqs. (11). Ordering the roots I_{1f}, I_{2f}, I_{3f} according to $I_{\min} < I_m < I_{\max}$ and introducing

$$I_{0f} = \tfrac{1}{2} a_f E_0^2 + I_{2f} , \tag{25}$$

the following conditions necessary and sufficient for $E_{f\pm}^2$ being real can be derived. If $a_f > 0, \Delta_f > 0$ hold, then

$$I_m < I_{0f} \leqslant I_{\max} \tag{26}$$

is necessary and sufficient for real $E_{f\pm}^2$. If $a_f < 0$, $\Delta_f > 0$ hold, then

$$I_{0f} < I_{\min} \tag{27}$$

or

$$I_m \leqslant I_{0f} < I_{\max} \tag{28}$$

is necessary and sufficient. If $a_f < 0$, $\Delta_f < 0$ the condition

$$I_{0f} < I_{2f} \tag{29}$$

must hold. $\Delta_f < 0$ and $a_f > 0$ is impossible, because Eq. (24) cannot be fulfilled [cf. Eq. (A6)].

Taking into account the previous CR [Eqs. (26)–(29)], we can specify the bounded and nonnegative field intensities. Due to the periodicity of $\wp$, we reduce the consideration in the following to the fundamental period parallelogram (FPP) [8] if $\Delta_f \neq 0$. Only the following cases are possible (cf. Appendix).

If Re $\omega_f = \omega$, the field intensity can be written as [11]

$$E_{f\pm}^2(z) = \frac{2}{a_f} \left\{ I_{\max} - I_{2f} + \frac{(I_{\max} - I_m)(I_{\max} - I_{\min})}{\wp(i(\text{Im } \omega_f \pm k_0 z); g_{2f}, g_{3f}) - I_{\max}} \right\} . \tag{30}$$

Since $-\infty \leqslant \wp(i(\text{Im } \omega_f \pm k_0 z); g_{2f}, g_{3f}) \leqslant I_{\min}$, Eq. (30) represents bounded solutions. If and only if

$$a_f > 0 \quad \text{and} \quad I_{1f} > \max(I_{2f}, I_{3f}) \tag{31}$$

or

$$a_f < 0 \quad \text{and} \quad I_{2f} > I_{1f} \tag{32}$$

these solutions are nonnegative. If Re $\omega_f = 0$ $(\Delta_f > 0)$, $E_{f\pm}^2$ reads, according to Eq. (5),

$$E_{f\pm}^2 = \frac{2}{a_f} [\wp(i(\text{Im } \omega_f \pm k_0 z); g_{2f}, g_{3f}) - I_{2f}]. \tag{33}$$

Obviously, $E_{f\pm}^2$ is nonnegative if and only if $a_f < 0$, because the upper bound of $\wp(i(\text{Im } \omega_f \pm k_0 z); g_{2f}, g_{3f})$ is $I_{\min}$ [11].

The necessary and sufficient condition for $E_{f\pm}^2(z)$ to be bounded (CB) is (ω' denotes the imaginary half-period of $\wp$ if $\Delta_f > 0$)

$$0 < \text{Im } \omega_f \pm k_0 d < \frac{2\omega'}{i} , \tag{34}$$

due to the location of the poles of $\wp$ in the FPP [8]. If Re $\omega_f = \omega_2$, evaluation of Eq. (5) yields, using the addition formula for $\wp$,

$$E_{f\pm}^2(z) = \frac{2}{a_f} \frac{|I_{2f} - I_{1f}|^2}{\wp(i(\text{Im } \omega_f \pm k_0 z); g_{2f}, g_{3f}) - I_{2f}}, \tag{35}$$

and this is non-negative and bounded if and only if $a_f < 0$ and

$$|\text{Im } \omega_f \pm k_0 d| < |\omega_2'|, \tag{36}$$

because $\wp(\omega_2'; g_{2f}, g_{3f}) = I_{2f}$, where ω_2' is the imaginary half-period of $\wp$ if $\Delta_f < 0$. The physical content of the foregoing analysis can be summarized as follows. Real, nonnegative and bounded solutions to Eq. (3) [according to Eq. (5)] are given by (i) Eq. (30), if $\Delta_f > 0$ and $I_m < I_{0f} \leqslant I_{\max}$. If $a_f > 0$, $I_{\max} = I_{1f}$ must hold. If $a_f < 0$, $I_{\max} = I_{2f}$ must hold. In both cases $E_{f\pm}^2$ are bounded; (ii) Eq. (33) if $\Delta_f > 0$ and $I_{0f} < I_{\min}$ and $a_f < 0$. In this case $0 < \text{Im } \omega_f \pm k_0 d < (2\omega'/i)$ must hold for bounded $E_{f\pm}^2(z)$; (iii) Eq. (35), if $\Delta_f < 0$ and $I_{0f} < I_{2f}$ and $a_f < 0$. $E_{f\pm}^2(z)$ are bounded if $|\text{Im } \omega_f \pm k_0 d| < |\omega_2'|$.

The field intensities in the substrate and in the cladding are represented by Eqs. (16), if conditions (15), (21), and (22) hold. $E_{\nu\pm}^2$ are bounded if $a_\nu > 0$. If $a_\nu < 0$, only E_{s-}^2 and E_{c+}^2 are bounded.

If the parameters $a_\nu, \overline{\epsilon}_\nu, n, E_0^2, k_0 d$ do not fulfill the foregoing conditions no guided waves according to Eq. (2) exist. In particular, there are no guided waves if, irrespective of $k_0 d$, the parameters are such that $a_f > 0$ and $\Delta_f < 0$ hold.

Subject to the above CRNB we now specify those normalized thickness $k_0 d$ (depending on $a_\nu, \overline{\epsilon}_\nu, n, E_0^2$) that are associated to field intensities $E_{\nu\pm}^2(z)$ obeying the boundary conditions. As a result, we get necessary and sufficient conditions for the existence of guided waves.

III. SOLUTIONS OF THE DISPERSION RELATION

If the boundary conditions are satisfied, then, additionally to Eqs. (18), (19), and (20), the equation

$$E_{f\pm}^2(d)=\frac{2}{a_f}[\wp(\omega_f\pm ik_0d,g_{2f},g_{3f})-I_{2f}]=\frac{2}{a_f}\lambda_\pm \tag{37}$$

must be fulfilled [12], where [13]

$$\lambda_\pm=-\frac{\overline{\epsilon_f}-\overline{\epsilon_c}}{2(1-a_c/a_f)}\pm\sqrt{D}, \tag{38}$$

$$D=\left[\frac{\overline{\epsilon_f}-\overline{\epsilon_c}}{2(1-a_c/a_f)}\right]^2+\frac{a_fC_f}{2(1-a_c/a_f)}. \tag{39}$$

Thus, condition (22) reads $q_c^2>(a_c/a_f)\lambda_\pm$.

Subject to the CR of the previous section the imaginary part of $\wp(\omega_f\pm ik_0d;g_{2f},g_{3f})$ vanishes, so that the right-hand side of

$$\wp(\omega_f\pm ik_0d;g_{2f},g_{3f})=\lambda_\pm+I_{2f} \tag{40}$$

must be real. Hence,

$$D\geqslant 0 \tag{41}$$

must hold and, since $E_{f\pm}^2(d)$ is non-negative,

$$\operatorname{sgn}\lambda_\pm=\operatorname{sgn}a_f. \tag{42}$$

Equation (40) constitutes a compact representation of the DR. If some of the parameters a_ν, $\overline{\epsilon_\nu}$, n, E_0^2, and k_0d are prescribed, the rest of the parameters must be determined consistently with Eqs. (40)–(42). Since a_s, a_f, $\overline{\epsilon_s}$, $\overline{\epsilon_f}$, n, and E_0^2 are embedded within ω_f, g_{2f}, and g_{3f}, it is rather hopeless to determine one of these quantities by Eq. (40) analytically. In this case, Eq. (40) can serve for verifying the existence of solutions (real tupels $\{a_\nu,\overline{\epsilon_\nu},n,E_0^2,k_0d\}$) and testing the consistency numerically.

Nevertheless, due to the fact that ω_f, g_{2f}, g_{3f}, $\lambda_\pm$, and I_{2f} are independent of k_0d in Eq. (40), it is appropriate to solve Eq. (40) with respect to k_0d and find the associated CS as follows.

Formal inversion of Eq. (40) yields the mode equations [4]

$$ik_0d=\pm w_+-\omega_f+2M\omega+2N\omega', \tag{43a}$$

for E_{f+}^2 and

$$-ik_0d=\pm w_--\omega_f+2M\omega+2N\omega', \tag{43b}$$

for E_{f-}^2, with

$$w_\pm=\int_{\lambda_\pm+I_{2f}}^{\infty}\frac{dI}{\sqrt{4I^3-g_{2f}I-g_{3f}}}, \tag{44}$$

where 2ω and $2\omega'$ denote the (in general) complex periods of $\wp$ and M,N are integers. Since Eqs. (43) must hold for real and positive k_0d it is convenient to write Eqs. (43) as a system, [14]

$$\operatorname{Re}(\pm w_\pm-\omega_f+2M\omega+2N\omega')=0, \tag{45a}$$

and, for E_{f+}^2 and E_{f-}^2 respectively,

$$\operatorname{Im}(\pm w_+-\omega_f+2M\omega+2N\omega')=k_0d, \tag{45b}$$

$$-\operatorname{Im}(\pm w_--\omega_f+2M\omega+2N\omega')=k_0d, \tag{45c}$$

that must hold for certain M,N.

To find the CS for Eq. (40) represented in the form (43) (with k_0d real and positive) or Eqs. (45) thus the allowed k_0d, it is useful to consider the CR of the previous section, since the CS are analogous to the CR.

(i) $\Delta_f>0$, $\operatorname{Re}\,\omega_f=\omega$, $i\omega'\in I\mathbb{R}$. Without loss of generality we assume that $\omega+i(\operatorname{Im}\,\omega_f\pm k_0d)\in$ FPP. Since function $\wp(\omega+i(\operatorname{Im}\,\omega_f\pm k_0d);g_{2f},g_{3f})$ decreases monotonically from $I_{\max}$ to I_m if $\operatorname{Im}\,\omega_f\pm k_0d\in[0,\omega'/i]$ and increases monotonically from I_m to $I_{\max}$ if $\operatorname{Im}\,\omega_f\pm k_0d\in[\omega'/i,2\omega'/i]$, each value $\wp\in[I_m,I_{\max}]$ is taken twice if $\operatorname{Im}\,\omega_f\pm k_0d\in[0,2\omega'/i]$. Thus, the CS is

$$I_m\leqslant\lambda_\pm+I_{2f}\leqslant I_{\max}, \tag{46}$$

which is identical to condition (26) with $\frac{1}{2}a_fE_0^2$ replaced by $\lambda_\pm$. Equations (45) read

$$\operatorname{Re}(\pm w_\pm-\omega+2M\omega)=0, \tag{47a}$$

$$\operatorname{Im}(\pm w_+-\omega_f+2N\omega')=k_0d, \tag{47b}$$

$$-\operatorname{Im}(\pm w_--\omega_f+2N\omega')=k_0d. \tag{47c}$$

Comparing Eq. (46) with Eqs. (26) and (28) it is obvious that Eq. (46) is equivalent to

$$\operatorname{Re}\,w_\pm=\omega. \tag{48}$$

Hence $M=0$ or $M=1$ is necessary in Eq. (47a) and the positive normalized thicknesses are given by Eqs. (47b) and (47c) according to

$$k_0d=\begin{cases}\operatorname{Im}(w_+-\omega_f), & \text{if } \lambda_+>\frac{1}{2}a_fE_0^2\\ \operatorname{Im}(w_+-\omega_f+2N\omega'), & \text{if } N=1,2,3,\ldots\\ \operatorname{Im}(-w_+-\omega_f+2N\omega'), & \text{if } N=2,3,4,\ldots,\end{cases} \tag{49}$$

associated with E_{f+}^2, and

$$k_0d=\begin{cases}\operatorname{Im}(-w_-+\omega_f), & \text{if } \frac{1}{2}a_fE_0^2>\lambda_-\\ \operatorname{Im}(-w_-+\omega_f+2N\omega'), & \text{if } N=1,2,3,\ldots\\ \operatorname{Im}(w_-+\omega_f+2N\omega'), & \text{if } N=-1,0,1,2,3,\ldots,\end{cases} \tag{50}$$

associated with E_{f-}^2 and subject to Eqs. (46), (31) or (32), and without further restrictions with respect to k_0d since $E_{f\pm}^2$ are bounded if $\operatorname{Re}\,\omega_f=\omega$.

(ii) $\Delta_f>0$, Re $\omega_f=0$, $\omega\in I\mathbb{R}$, $i\omega'\in I\mathbb{R}$. We assume that $i(\text{Im }\omega_f\pm k_0d)\in$ FPP. In this case $\wp(i(\text{Im }\omega_f\pm k_0d);g_{2f},g_{3f})$ increases monotonically from $-\infty$ to $I_{\min}$ and then decreases monotonically from $I_{\min}$ to $-\infty$ as Im $\omega_f\pm k_0d\in[0,2\omega'/i]$. Hence we get the necessary and sufficient CS

$$\lambda_\pm+I_{2f}\leqslant I_{\min} \tag{51}$$

which is equivalent to

$$\text{Re } w_\pm=0, \tag{52}$$

in analogy to the equivalence of Re $\omega_f=0$ and condition (27). Similar considerations as in the previous case (i) lead to $M=0$ in Eq. (45a) and thus to the possible mode equations

$$k_0d=\begin{cases}\text{Im}(\pm w_+-\omega_f+2N\omega') & \text{for } E_{f+}^2\\ -\text{Im}(\pm w_--\omega_f+2N\omega') & \text{for } E_{f-}^2.\end{cases} \tag{53}$$

Evaluating the CB [Eq. (34)] the positive normalized thicknesses associated to bounded E_{f+}^2 are thus

$$k_0d=\begin{cases}\text{Im}(w_+-\omega_f) & \text{if } \lambda_+>\frac{1}{2}a_fE_0^2\\ \text{Im}(-w_+-\omega_f+2\omega').\end{cases} \tag{54}$$

E_{f-}^2 is bounded if the thickness is given by

$$k_0d=\text{Im}(-w_-+\omega_f), \tag{55}$$

subject to $\lambda_-<\frac{1}{2}a_fE_0^2$.

(iii) $\Delta_f<0$, Re $\omega_f=\omega_2$, $\omega_2\in I\mathbb{R}, i\omega_2'\in I\mathbb{R}$. The primitive periods of $\wp$ are $2\omega_2$ and $\omega_2+\omega_2'$ [15]. Assuming $\omega_2+i(\text{Im }\omega_f\pm k_0d)\in$ FPP, in this case $\wp(\omega_2+i(\text{Im }\omega_f\pm k_0d);g_{2f},g_{3f})=\lambda_\pm+I_{2f}$ must be solved for k_0d. For Im $\omega_f\pm k_0d\in[-\omega_2'/i,0]$, $\wp(\omega_2+i(\text{Im }\omega_f\pm k_0d);g_{2f},g_{3f})$ increases monotonically from $-\infty$ to I_{2f} and then decreases monotonically from I_{2f} to $-\infty$ for $(\text{Im }\omega_f\pm k_0d)\in[0,+\omega_2'/i]$. Hence, we get the necessary and sufficient CS

$$\lambda_\pm\leqslant 0, \tag{56}$$

equivalent to

$$\text{Re } w_\pm=\omega_2. \tag{57}$$

Equations (45) read [16]

$$\text{Re}(\pm w_\pm-\omega_f+(2M+N)\omega_2)=0 \tag{58a}$$

$$\pm\text{Im}(\pm w_\pm-\omega_f+N\omega_2')=k_0d. \tag{58b}$$

Hence, $2M+N=0$ or $2M+N=2$ is necessary and the positive thicknesses associated with bounded E_{f+}^2 are

$$k_0d=\begin{cases}\text{Im}(w_+-\omega_f) & \text{if } \frac{1}{2}a_fE_0^2<\lambda_+,\\ \text{Im}(-w_+-\omega_f)\end{cases} \tag{59}$$

if $|\text{Im }\omega_f+k_0d|<|\omega_2'|$ is evaluated. If

$$k_0d=\text{Im}(-w_-+\omega_f) \quad \text{and} \quad \frac{1}{2}a_fE_0^2>\lambda_-, \tag{60}$$

E_{f-}^2 is bounded.

In order to find physical solutions $E_{\nu\pm}^2$, $\nu=s,f,c$, the normalized thickness k_0d must be given by Eqs. (49), (50), (54), (55), (59), and (60) (for E_{f+}^2 and E_{f-}^2, respectively) and then the appropriate $E_{\nu\pm}^2$, $\nu=s,c$ (satisfying the CRNB) must be matched with E_{f+}^2 or E_{f-}^2 (or both) at the boundaries. In general, the appropriate combination of signs in the sequence $E_{s\pm}^2$, $E_{f\pm}^2$, and $E_{c\pm}^2$ conveniently can be found by using

$$\begin{aligned}\text{sgn}\left(\frac{dE_{s\pm}^2}{dz}\bigg|_{z=0}\right)&=\pm\ \text{sgn } a_s,\\ \text{sgn}\left(\frac{dE_{c\pm}^2}{dz}\bigg|_{z=d}\right)&=\pm\ \text{sgn } a_c,\\ \frac{dE_{f\pm}^2}{dz}\bigg|_{z=0,d}&=\pm\frac{2k_0|\wp'|}{a_f}\,i\epsilon\bigg|_{z=0,d},\end{aligned} \tag{61}$$

where $\wp'=d\wp(u;g_2,g_3)/du$ and $\epsilon=(\wp'/|\wp'|)$.

For $u=\text{Re }\omega_f+i(\text{Im }\omega_f\pm k_0z)$, Re $\omega_f=0,\omega,\omega_2$, ϵ is given by Fig. 2 [16]. Obviously, the sign of $dE_{f\pm}^2/dz$ at $z=0,d$ depends on the magnitude of Im ω_f and of Im $\omega_f\pm k_0d$ and thus on the appropriate mode equations (43). Evaluation is facilitated by using the associated inequalities for Im ω_f and Im $w_\pm$ (cf. Appendix). An example is given in the next chapter.

With reference to the summary at the end of the previous section the results of this section can be summarized. In order to match the field intensities at the boundaries according to (i)–(iii) of Sec. II [Eqs. (16), (30), (33), and (35)] the following necessary and sufficient conditions must hold additionally: $I_m\leqslant\lambda_\pm+I_{2f}\leqslant I_{\max}$ for (i), $\lambda_\pm+I_{2f}\leqslant I_{\min}$ for (ii), $\lambda_\pm\leqslant 0$ for (iii). The associated normalized thicknesses are given by Eqs. (49),(50) for (i), by Eqs. (54),(55) for (ii), by Eqs. (50),(60) for (iii), respectively. Inserting the appropriate positive k_0d into Eqs. (16), (30), (33), and (35), respectively, and making use of Eqs. (61) the possible combinations $E_{s\pm}^2$, $E_{f\pm}^2$, $E_{c\pm}^2$ are obtained. In this way all realizable guided waves (2) as well as necessary and sufficient conditions for their existence (CRNB and CS) are found.

It should be noted that the k_0d obeying the DR can be expressed by elliptic integrals whereas the CRNS are algebraic, only containing $\Delta_f,I_{0f},I_{1f},I_{2f},I_{3f}$. Thus, these conditions can easily be evaluated [cf. (i) and (ii) of Sec. IV].

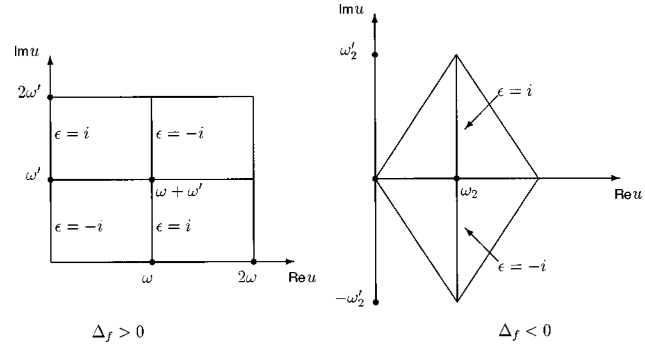

FIG. 2. Fundamental period parallelogram (FPP) for positive and negative discriminants Δ_f. $\epsilon=\wp'/|\wp'|$ denotes the normalized derivative of Weierstrass's function $\wp(u)$.

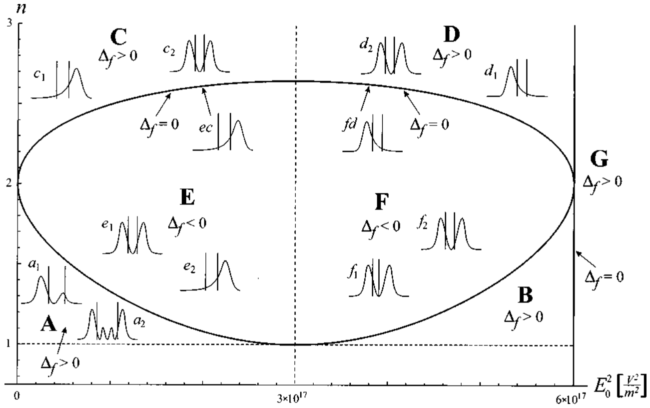

FIG. 3. Subsets A–G (discriminated by Δ_f) of the (E_0^2,n) plane with patterns of physical field intensities (unscaled).

The customary way to investigate the DR (40) (cf. [1,4]) is to prescribe the normalized thickness k_0d and the material parameters a_ν, $\overline{\epsilon_\nu}$, $\nu=s,f,c$ and then to solve Eq. (40) numerically with respect to n, E_0^2. This procedure works if and only if the CRNS are satisfied for a certain domain in the (E_0^2,n) plane. If so, k_0d, according to Eqs. (49), (50), (54), (55), (59), and (60) can be plotted as a function of E_0^2,n and then it can be checked whether or not the prescribed value k_0d is taken by this function.

In principle, this procedure can be used for any subset of the set of tupels $\{a_\nu,\overline{\epsilon_\nu},n,E_0^2,k_0d\}$, $\nu=s,f,c$ if certain parameters a_ν, $\overline{\epsilon_\nu}$ $(\nu=s,f,c)$, n, E_0^2, and k_0d are prescribed.

IV. APPLICATIONS

A. A numerical example

To illustrate the above results we choose the material parameters as in Ref. [1] $a_s=a_c=0$, $a_f=\pm 10^{-17}(\mathrm{m}^2/V^2)$, $\overline{\epsilon_f}=4$, $\overline{\epsilon_s}=\overline{\epsilon_c}=1$ but with $a_s=a_c=10^{-25}\,\mathrm{m}^2/\mathrm{V}^2$ instead of $a_s=a_c=0$. A negative Kerr coefficient $a_f=-10^{-17}(\mathrm{m}^2/\mathrm{V}^2)$ was selected because the case $a_f>0$ is rather simple, since the condition $a_f>0$ and $\Delta_f<0$ is not consistent with Eq. (24). Thus $E_{f\pm}^2$ are always bounded if $a_f>0$, because Re $\omega_f=\omega$ in this case [cf. Eq. (A2)]. To find the parameters E_0^2, n, k_0d associated to physical solutions $E_{\nu\pm}^2(z)$ the following procedure seems appropriate.

(i) *Verifying CRN*. With the above prescribed parameters evaluation yields $D\geqslant 0$ if $a_fE_0^2<0$, $\lambda_+<0$ if $-6<a_fE_0^2<0$, $\lambda_-<0$ if $a_fE_0^2\leqslant 0$, so that conditions (21) and (22) combined with Eqs. (37) and (41) define subsets of the (E_0^2,n) plane (see Fig. 3) by

$$n^2>\overline{\epsilon_s}+\tfrac{1}{2}a_sE_0^2 \quad \text{if} \;\; a_fE_0^2<0 \tag{62a}$$

and

$$n^2>\overline{\epsilon_c}+\frac{a_c}{a_f}\lambda_+ \quad \text{if} \;\; -6<a_fE_0^2<0 \tag{62b}$$

and

$$n^2>\overline{\epsilon_c}+\frac{a_c}{a_f}\lambda_- \quad \text{if} \;\; a_fE_0^2<0. \tag{62c}$$

In particular, this means that $E_{\nu+}^2$, $(\nu=s,c)$, must be excluded if $a_fE_0^2\leqslant -6$ (domain G). These conditions specify $\{E_0^2,n\}$ associated with real and nonnegative $E_{\nu\pm}^2$, $\nu=s,c$. If $\nu=f$ the conditions for the different regions designated in Fig. 3 are associated as follows (the equations for Re ω_f and Re $\omega_\pm$ in brackets represent the corresponding equivalent CR and CS):

$$\begin{gathered}(A)\;\; I_{3f}<I_{1f}\leqslant I_{0f}<I_{2f}\;(\mathrm{Re}\;\omega_f=\omega),\quad \Delta_f>0,\\ (B)\;\; I_{0f}\leqslant I_{3f}<I_{1f}<I_{2f}\;(\mathrm{Re}\;\omega_f=0),\;\; \Delta_f>0,\\ (C\vee D)\;\; I_{0f}<I_{2f}<I_{3f}<I_{1f}\;(\mathrm{Re}\;\omega_f=0),\quad \Delta_f>0,\\ (E\vee F)\;\; I_{0f}<I_{2f}\;(\mathrm{Re}\;\omega_f=\omega_2),\quad \Delta_f<0,\\ (G)\;\; I_{0f}<I_{3f}\leqslant I_{2f}<I_{1f}\;(\mathrm{Re}\;\omega_f=0),\;\; \Delta_f>0.\end{gathered} \tag{63}$$

As follows from Eqs. (A3)–(A10), the field intensities $E_{f\pm}^2(z)$ corresponding to the different subsets of the (E_0^2,n) plane, given by Eqs. (30), (33), and (35) are all real and nonnegative.

(ii) *Verifying CS and solving DR with respect to* k_0d. Due to the special choice of the parameters [$\overline{\epsilon_\nu}$ and a_ν are given, so that Eq. (40) must not be solved with respect to these parameters] and subject to the corresponding CS,

$$\begin{gathered}(A)\;\; I_{3f}<I_{1f}\leqslant\lambda_+ +I_{2f}<I_{2f}\;(\mathrm{Re}\;w_+=\omega),\\ (B)\;\; \lambda_- +I_{2f}\leqslant I_{3f}<I_{1f}<I_{2f}\;(\mathrm{Re}\;w_-=0),\\ (C\vee D)\;\; \lambda_\pm +I_{2f}<I_{2f}<I_{3f}<I_{1f}\;(\mathrm{Re}\;w_\pm=0),\\ (E\vee F)\;\; \lambda_\pm<0\;\;(\mathrm{Re}\;w_\pm=\omega_2),\\ (G)\lambda_- +I_{2f}<I_{3f}<I_{2f}<I_{1f}\;(\mathrm{Re}\;w_-=0),\end{gathered} \tag{64}$$

the DR (40) can be solved for k_0d yielding the allowed thicknesses.

(iii) *Determining the positive thicknesses* k_0d *associated with field intensities* $E_{f\pm}^2$ *obeying CRNB*. Since the $E_{f\pm}^2$ are = bounded if Re $\omega_f=\omega$ only the CB (34),(36) must be evaluated subject to $k_0d>0$. We obtain the following results:

(A) E_{f+}^2 is bounded if and only if

$$k_0d=\begin{cases}\mathrm{Im}(w_+-\omega_f+2N\omega'), & N=1,2,3,\ldots\\ \mathrm{Im}(-w_+-\omega_f+2N\omega'), & N=2,3,4,\ldots;\end{cases}$$

(B) bounded E_{f-}^2 do not exist, since $k_0 d=0$ according to Eq. (55);

(C) E_{f+}^2 is bounded if and only if

$$k_0 d=\mathrm{Im}(-w_+-\omega_f+2\omega'),$$

E_{f-}^2 is bounded if and only if

$$k_0 d=\mathrm{Im}(-w_-+\omega_f) \quad \text{and} \quad \lambda_-<\tfrac{1}{2}a_f E_0^2,$$

(D) E_{f+}^2 is bounded if and only if

$$k_0 d=\begin{cases}\mathrm{Im}(w_+-\omega_f) \quad \text{and} \quad \lambda_+>\tfrac{1}{2}a_f E_0^2\\ \mathrm{Im}(-w_+-\omega_f+2\omega'),\end{cases} \tag{65}$$

(E) E_{f+}^2 is bounded if and only if

$$k_0 d= \mathrm{Im}(-w_+-\omega_f),$$

E_{f-}^2 is bounded if and only if

$$k_0 d=\mathrm{Im}(-w_-+\omega_f) \quad \text{and} \quad \lambda_-<\tfrac{1}{2}a_f E_0^2,$$

(F) E_{f+}^2 is bounded if and only if

$$k_0 d=\begin{cases}\mathrm{Im}(w_+-\omega_f) \quad \text{and} \quad \lambda_+>\tfrac{1}{2}a_f E_0^2,\\ \mathrm{Im}(-w_+-\omega_f),\end{cases}$$

(G) bounded E_{f+}^2 do not exist.

(iv) *Finding the appropriate combination of signs in the sequence $E_{s\pm}^2, E_{f\pm}^2, E_{c\pm}^2$ obeying CRNB.* Using Eqs. (49), (50), (54), (55), (59), and (60), the normalized thicknesses given in the foregoing section are associated with $E_{f\pm}^2$ that can be combined with $E_{s\pm}^2$ and $E_{c\pm}^2$ as follows [labels a, c, d, e, and f refer to Fig. 3]:

$$\text{(A)} \quad k_0 d=\begin{cases}\mathrm{Im}(w_+-\omega_f+2N\omega'), \quad N=1,2,3,\ldots;\ E_{s-}^2,\ E_{f+}^2,\ E_{c-}^2 \quad (a_1)\\ \mathrm{Im}(-w_+-\omega_f+2N\omega'), \quad N=2,3,4,\ldots;\ E_{s-}^2,\ E_{f+}^2,\ E_{c+}^2 \quad (a_2),\end{cases}$$

$$\text{(C)} \quad k_0 d=\begin{cases}\mathrm{Im}(-w_+-\omega_f+2\omega')\ ;\ E_{s-}^2,\ E_{f+}^2,\ E_{c+}^2 \quad (c_1)\\ \mathrm{Im}(-w_-+\omega_f);\ E_{s+}^2,\ E_{f-}^2,\ E_{c+}^2 \quad (c_2),\end{cases}$$

$$\text{(D)} \quad k_0 d=\begin{cases}\mathrm{Im}(w_+-\omega_f);\ E_{s-}^2,\ E_{f+}^2,\ E_{c-}^2 \quad (d_1)\\ \mathrm{Im}(-w_+-\omega_f+2\omega');\ E_{s-}^2,\ E_{f+}^2,\ E_{c+}^2 \quad (d_2),\end{cases} \tag{66}$$

$$\text{(E)} \quad k_0 d=\begin{cases}\mathrm{Im}(-w_+-\omega_f);\ E_{s-}^2,\ E_{f+}^2,\ E_{c+}^2 \quad (e_1)\\ \mathrm{Im}(-w_-+\omega_f);\ E_{s+}^2,\ E_{f-}^2,\ E_{c+}^2 \quad (e_2),\end{cases}$$

$$\text{(F)} \quad k_0 d=\begin{cases}\mathrm{Im}(w_+-\omega_f);\ E_{s-}^2,\ E_{f+}^2,\ E_{c-}^2 \quad (f_1)\\ \mathrm{Im}(-w_+-\omega_f);\ E_{s-}^2,\ E_{f+}^2,\ E_{c+}^2 \quad (f_2).\end{cases}$$

Boundary between (E) and (C): $k_0 d=\mathrm{Im}(-w_-+\omega_f);\ E_{s+}^2,\ E_{f-}^2,\ E_{c+}^2\ (ec)$.
Boundary between (F) and (D): $k_0 d=\mathrm{Im}(w_+-\omega_f);\ E_{s-}^2,\ E_{f+}^2,\ E_{c-}^2\ (fd)$.

To sum up, if the parameters a_ν and $\overline{\epsilon_\nu}$ are given, physical solutions $E_{\nu\pm}^2$ can be found by choosing the parameters n, E_0^2, and $k_0 d$ appropriately, as shown in Fig. 3. In particular, some results can be compared with those of the linear case. Obviously, Eqs. (62) are generalizations of $n^2>\max(\overline{\epsilon_s},\overline{\epsilon_c})$ [6,7]. They specify, in a manner that is different for E_{f+}^2 and E_{f-}^2, a cutoff with respect to the effective wave index n depending on the intensity E_0^2. The analogon of the linear-case condition $n^2<\overline{\epsilon_f}$ [6,7] seems to be the conditions of (i)–(iii) in Sec. II (without the CB containing $k_0 d$). Remarkably, these conditions depending on the sign of a_f and of the discriminant Δ_f can be expressed in terms of the roots I_{1f}, I_{2f}, I_{3f} according to Eq. (11). In addition to these conditions the CB lead to specific exclusions [domains (B),(G)] not appearing in the linear case because $n^2<\overline{\epsilon_f}$ implies bounded field intensities in the linear film [6] (it would be intriguing to investigate whether the occurrence of the cutoff at $E_0^2=6\times10^{17}$ is specific for the parameters chosen or whether it is a general consequence of the CB). Equations (66) show that the CRNB and the CS specify two mode equations for k_0d with E_0^2, n in the allowed domains (A),(C),(D),(E),(F). This means that exactly two (in general, different) normalized thicknesses k_0d are determined for each E_0^2,n. Since the mode equations define k_0d as a continuous function of E_0^2,n in each of the allowed domains, there must be a lower and upper bound of k_0d for the corresponding domain. Thus a lower and an upper cutoff with respect to k_0d can be determined numerically (for the parameters $a_\nu\overline{\epsilon_\nu}$ chosen and for the corresponding domain in the (E_0^2,n plane). It seems that these results could be of practical importance due to their potential use in designing optical waveguides.

Finally, it should be noted that we included the limiting case $\Delta_f=0$ in Fig. 3. It is rather interesting, that the (unscaled) intensity patterns (ec and c_1, fd and d_1) are the same though the corresponding mode equations in Eqs. (66) are different.

B. On the origin of singular field intensities

As the second application, we address the question: Is it necessary to explain the occurrence of singular field intensities by neglecting absorption within the nonlinear media? It seems to be nontrivial to extend the foregoing analysis to include absorption. Apparently, an analytical solution to the nonlinear Helmholtz equation for absorbing dielectric layers is not known in the literature. Nevertheless, it has been argued that singularities of electric fields ''can be weakened by the presence of damping'' [17]. Physical intuition is not very reliable in nonlinear problems, so that it seems appropriate to try a different explanation of the occurrence of singular fields.

Considering the example in Ref. [1] ($a_s=a_c=0$, $a_f=-10^{-17}$ m$^2/V^2$, $\overline{\epsilon_f}=4$, $\overline{\epsilon_s}=\overline{\epsilon_c}=1$, $a_fE_0^2=-0.5$, and $n=1.589$, cf. [4]), we obtain $\Delta_f<0$ and

$$\text{Re } \omega_f = \text{ Re } w_\pm=\omega_2=1.749,$$

$$\omega_2=3.141i,$$

$$\text{Im } \omega_f=\text{Im } w_+=-0.665,$$

$$\text{Im } w_-=-2.476, \quad \lambda_\pm<0.$$

According to Eqs. (59) and (60), the following possibilities remain:

$$k_0d=\text{Im}(-w_+-\omega_f)=1.331 \tag{67}$$

for bounded E_{f+}^2 and

$$k_0d=\text{Im}(-w_-+\omega_f)=1.810 \tag{68}$$

for bounded E_{f-}^2. E_{f+}^2 must be matched with E_{s-}^2 and E_{c+}^2, both being unbounded. E_{f-}^2 must be matched with E_{s+}^2 and E_{c+}^2, E_{c+}^2 being unbounded. Thus, there is no combination of the field intensities $E_{\nu\pm}^2$ with all $E_{\nu\pm}^2$ being bounded.

If $k_0d=\pi$ ($E_0^2=5\times10^{16}$, $n=1.5887$) is selected, evaluation of Im ω_f+k_0d in Eq. (36) yields bounded E_{f+}^2 and unbounded E_{f-}^2 [since Eq. (36) is not satisfied] in agreement with Ref. [1]. The field intensities E_{s-}^2, E_{f+}^2, and E_{c+}^2 fit at the boundaries, but E_{s-}^2 and E_{c+}^2 are not bounded. E_{f-}^2 fits with E_{s+}^2, and E_{c-}^2 (both are bounded). But, according to Eqs. (67) and (68), $k_0d=\pi$ is not the appropriate thickness. Changing slightly the parameters a_s and a_c ($a_s=a_c=-0.2a_f$), we obtain, using Eqs. (67) and (68), that $k_0d=1.346$ for E_{f+}^2 and $k_0d=1.769$ for E_{f-}^2. E_{f+}^2 remains bounded and now fits with bounded E_{s-}^2 and E_{c+}^2. E_{f-}^2 is bounded if $k_0d=1.769$ and fits with bounded E_{s+}^2 and E_{c+}^2 in this case. Thus, we have demonstrated that by changing real parameters a_s and a_c of the substrate and of the cladding and selecting the appropriate thickness $k_0d=1.769$, one can remove the singularity of E_{f-}^2.

It may be that a singularity of the field intensity is an artifact due to the use of a real local permittivity. But this has not yet been proved analytically. On the other hand, the assumed local Kerr-like real permittivity is not unphysical, since it gives rise to well-known phenomena in nonlinear optics.

To conclude, the solutions obtained according to Eqs. (5) are generally singular, and it is necessary to restrict the choice of parameters by certain conditions given above to avoid singular solutions. As shown above [cf. Eqs. (A1)-(A6)], there are no singular solutions at all if the Kerr coefficients a_ν are positive. In particular, if $a_f<0$ and $\Delta_f<0$, the thickness k_0d must be determined appropriately with parameters a_ν, $\overline{\epsilon_\nu}$, n, and E_0^2 according to CNR and Eqs. (58) and (36). Thus, a possible explanation for the existence of singular field intensities is as follows: the thickness d of the film is not appropriate.

C. Guided waves in a structure with linear substrate and film and nonlinear cladding

As a third application the case $a_s=a_f=0$, $a_c>0$ [17,18] is briefly discussed. If $a_f=0$, the discriminant Δ_f vanishes, this case has been excluded in the above analysis. A basic assumption of Sec. II was $a_f\neq0$. If $a_f=0$ some of the results of Sec. II become meaningless. For example, ω becomes infinite in the CR Eq. (23) and in the CS Eq. (48) if $a_f=0$, $q_f^2<0$, since $I_{1f}=I_{2f}$ in this case. Hence, ω_f in Eq. (37) is not defined. It seems rather involved to evaluate the results of Sec. II in the limit $a_f\to0$. Thus it is appropriate to go back to Eq. (3) and solve it subject to the constraints (13), (15), (21), and (2). Hence

$$E_s(z)=E_0e^{q_sk_0z}, \quad z\leq0 \tag{69}$$

$$E_f(z)=E_0\left(\cosh q_fk_0z+\frac{q_s}{q_f}\sinh q_fk_0z\right), \quad 0\leq z\leq d, \tag{70}$$

$$E_{c\pm}(z)=\sqrt{\frac{2q_c^2}{a_c}}\frac{1}{\sin[\sqrt{q_c^2}(\omega_c\pm ik_0z)]},\quad z\geqslant d, \tag{71}$$

where ω_c is given by Eq. (19). According to Sec. II both solutions $E_{c\pm}(z)$ are bounded. Using Eq. (19) the continuity of dE/dz at $z=d$ can be evaluated leading to (cf. Refs. [17,18])

$$\tanh q_f k_0 d+\frac{q_f(q_s\mp\tilde{q}_c)}{q_f^2\mp q_s\tilde{q}_c}=0, \tag{72}$$

where the upper and the lower signs refer to E_{c+} and E_{c-}, respectively. The nonlinear dispersion relation (72) resembles the linear one with the exception that $\tilde{q}_c=\sqrt{q_c^2-(a_c/2)E_f^2(d)}$ was introduced. According to Eq. (22) $\tilde{q}_c$ is real. $\tilde{q}_c$ is equal to q_c if $a_c=0$ and, since only $E_{c-}(z)$ is consistent with Eq. (12) in this case, Eq. (72) is reduced to the linear dispersion relation [20]. Equation (72) can be written as

$$\frac{a_c}{2}E_0^2=\frac{q_c^2\left(\cosh q_f k_0 d+\dfrac{q_s}{q_f}\sinh q_f k_0 d\right)^2-(q_f\sinh q_f k_0 d+q_s\cosh q_f k_0 d)^2}{\left(\cosh q_f k_0 d+\dfrac{q_s}{q_f}\sinh q_f k_0 d\right)^4}. \tag{73}$$

This version may be useful for finding parameters $\{a_c,\overline{\epsilon_\nu},n,E_0^2,k_0d\}$ compatible with the nonlinear dispersion relation.

V. CONCLUSION

TE-polarized waves according to Eqs. (2) and (12) supported by a lossless Kerr-like nonlinear three-layer structure have been investigated. Necessary and sufficient conditions for the existence of real, non-negative, and bounded field intensities and a general dispersion relation have been obtained. It has been shown that, subject to certain necessary and sufficient solvability conditions, the dispersion relation has solutions (certain domains in the parameter space $\{a_\nu,\overline{\epsilon_\nu},n,E_0^2,k_0d\}$ that are consistent with the dispersion relation). In particular, the dispersion relation can be solved with respect to the normalized thickness k_0d if the solvability conditions are satisfied. All k_0d consistent with the dispersion relation and associated with real, non-negative, and bounded field intensities are specified. Thus it seems that the description of TE waves guided by a lossless Kerr-like nonlinear three-layer structure has been brought to a certain close. Unsolved problems refer to absorbing layers and a stability analysis of the guided waves.

ACKNOWLEDGMENT

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APPENDIX: EXPRESSIONS FOR ω_ν, $\nu=s,f,c$

Subject to the constraints (21)–(24) evaluation of Eqs. (6) yields

$$\omega_s=\begin{cases}\dfrac{1}{\sqrt{q_s^2}}\left(\dfrac{\pi}{2}+i\ln(\alpha+\sqrt{\alpha^2-1})\right), & a_s>0,\quad \alpha=\sqrt{\dfrac{2q_s^2}{a_sE_0^2}}>1\\[2ex] \dfrac{i}{\sqrt{q_s^2}}\ln\left(\sqrt{\dfrac{2q_s^2}{-a_sE_0^2}}+\sqrt{1-\dfrac{2q_s^2}{a_sE_0^2}}\right), & a_s<0,\end{cases} \tag{A1}$$

$$\omega_c\pm ik_0d=\begin{cases}\dfrac{1}{\sqrt{q_c^2}}\left[\dfrac{\pi}{2}+i\ln\left(\sqrt{\dfrac{a_fq_c^2}{a_c\lambda_\pm}}+\sqrt{\dfrac{a_fq_c^2}{a_c\lambda_\pm}-1}\right)\right], & a_c>0\\[2ex] \dfrac{i}{\sqrt{q_c^2}}\ln\left(\sqrt{\dfrac{a_fq_c^2}{-a_c\lambda_\pm}}+\sqrt{1-\dfrac{a_fq_c^2}{a_c\lambda_\pm}}\right), & a_c<0,\end{cases} \tag{A2}$$

$$\omega_f=\omega+\int_{I_{0f}}^{I_{\max}}\frac{dt}{\sqrt{4t^3-g_{2f}t-g_{3f}}}+2\omega',\quad \text{if } \Delta_f>0,\ a_f>0,\quad I_m<I_{0f}\leqslant I_{\max}, \tag{A3}$$

$$\omega_f = \omega' + \int_{I_{0f}}^{I_{\min}} \frac{dt}{\sqrt{4t^3 - g_{2f}t - g_{3f}}}, \quad \text{if } \Delta_f > 0,\ a_f < 0, \quad I_{0f} < I_{\min}, \tag{A4}$$

$$\omega_f = \omega + \int_{I_{0f}}^{I_{\max}} \frac{dt}{\sqrt{4t^3 - g_{2f}t - g_{3f}}} + 2\omega', \quad \text{if } \Delta_f > 0,\ a_f < 0,\ I_m \leqslant I_{0f} < I_{\max}, \tag{A5}$$

$$\omega_f = \omega_2 + \int_{I_{0f}}^{I_{2f}} \frac{dt}{\sqrt{4t^3 - g_{2f}t - g_{3f}}}, \quad \text{if } \Delta_f < 0,\ a_f < 0,\ I_{0f} < I_{2f}. \tag{A6}$$

Due to conditions (46), (51), and (56), $w_\pm$ is given by the above expression for ω_f if $\frac{1}{2}a_f E_0^2$ is replaced by $\lambda_\pm$, respectively. The above expressions ω_f imply, in particular, since

$$\omega' = \int_{I_m}^{I_{\max}} \frac{dt}{\sqrt{4t^3 - g_{2f}t - g_{3f}}},$$

$$\operatorname{Im} \omega_f \geqslant \operatorname{Im} \omega' > 0, \tag{A7}$$

if $a_f < 0$, $I_m \leqslant I_{0f} < I_{\max}$ and

$$\operatorname{Im} \omega_f > \operatorname{Im} \omega' \geqslant 0, \tag{A8}$$

if $a_f > 0$, $I_m < I_{0f} \leqslant I_{\max}$, and, since

$$\omega' = i \int_{\infty}^{I_{\min}} \frac{dt}{\sqrt{|4t^3 - g_{2f}t - g_{3f}|}},$$

$$0 < \operatorname{Im} \omega_f < \operatorname{Im} \omega', \tag{A9}$$

if $a_f < 0$ and $I_{0f} < I_{\min}$. Finally, since

$$\omega_2' = i \int_{-\infty}^{I_{2f}} = \frac{dt}{\sqrt{|4t^3 - g_{2f}t - g_{3f}|}},$$

$$|\operatorname{Im} \omega_f| < \operatorname{Im} |\omega_2'|, \tag{A10}$$

if $a_f < 0$ and $I_{0f} < I_{2f}$. The same conditions hold, if ω_f is replaced by $w_\pm$ and I_{0f} is replaced by $\lambda_\pm + I_{2f}$.

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